Stability analysis for switched systems*

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Abstract. In this paper, we treat the Hurwitz stability criterion for a convex fuzzy set of matrices represented by autonomous linear time invariants systems. We give necessary and sufficient conditions to assure the Hurwitz conditions in order to attest the existence of a common quadratic Lyapunov function in an arbitrary set of matrices, associated by a fuzzy set.

Keywords: Switched systems, Lyapunov functions, convex fuzzy sets, Hurwitz matrix, pencil of matrices

1 Introduction

This paper discusses the problem of find the negative roots of a set of matrices, in order to assure the stability conditions for a stable performance of a switched system ΣA_z by a pair of linear subsystems ΣA_i and ΣA_j. We consider the problem of verifying the stability of a convex set K of matrices, more specifically in an interval matrix. These results are motivated by the fact of obtain a stable performance of the fuzzy switched systems, minimize the energy dissipated and to preserve the stability properties in all the switched actions [12], [2], [9], [4]. We assure the stable performance for the convex set of matrices in its pencil. The matrices treated in the pencil, belong to linear time-invariant systems $\dot{x}(t) = Ax(t)$ where the parametric matrix $A \in K$, and the system resultant after the switched actions, is considered a linear time-variant system $\dot{x}(t) = A_z x(t)$. In this paper, we give necessary and sufficient conditions in K.

By a switched system, we mean a hybrid dynamical system consisting of a family of continuous-time subsystems and rules that orchestrates the switching between them.

Switching systems have numerous applications in control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters and many others fields. In the last few years, every major control conference has had several regular and invited sessions on switching systems and control [7], [10].

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We assume that the states of the Switched Fuzzy System (SFS), does not jump at the switching instant, i.e., the solution $x(\cdot)$ is every where continuous. The systems ΣA_i and ΣA_j are continuous, and in the Switching Action (SA) there are interrupted by switching signals then, the switched system resulting is every where continuous.

The stability conditions for the pencil of matrices, are not in terms of the Lyapunov equation, there are exposed by Hurwitz stability criterion. This criterion is useful to establish fundaments to demonstrate the existence of a common quadratic Lyapunov function (CQLF) in an arbitrary compact convex sets, including the interval of matrices. A general result is given for matrices of n order.

Here, we do not treat extensively with the continuous fuzzy systems, we only utilize some notation and ideas. For more information about fuzzy systems, see [16], [12].

This paper is organized as follows, in section 2, we give some mathematical preliminaries and definitions. In section 3 , we present the principal results by a set of theorems and, finally the section 4 is about the conclusions.

2 Preliminaries

Consider the dynamical system

$$\dot{x} = Ax, \qquad A \in \mathcal{K} \triangleq \{A_1, A_2, ..., A_m\} \tag{1}$$

where A_i, $i=\{1,2,...,m\}$, are constant matrices in R^{nxn}. The matrices A_i, are assumed Hurwitz[1]. An important problem is to determine necessary and sufficient conditions for assure that a pencil of Hurwitz matrices nxn is Hurwitz in its interval (the SFS ΣA_z resultant are for an arbitrary switching sequence Hurwitz). This result is showed in the theorems 5 and 6 for the case 2x2 and nxn respectively. Hence, the existence of such conditions are sufficient to guarantee the uniform asymptotic (exponential) stability of the switching system, compose by elements of the set A.

Definition 1. The switched matrix is:

$$A_z(t) = \left(\frac{w_i A_i + w_j A_j}{w_i + w_j}\right) \tag{2}$$

where $i \neq j$ and $i,j \in \{1,...,m\}$, m is the number of systems in the phase plane, $x \in R^{nx1}$ is the state vector, and the matrices $A_i \in R^{nxn}$ each linear component is called subsystem, furthermore $w_i=(1-w_i)$. We can see from (2) that it is possible to represent a switched fuzzy system like Takagi-Sugeno a fuzzy model [5], [16], then

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[1] The eigenvalues of each A_i matrix lies in the open left half of the complex plane, denoted C^-; at times, for clarity, we also refer to this as an asymptotically stable matrix.

$$\dot{x}(t) = \left(\frac{\sum_{z=i,(j)}^{j,(i)} w_z\ A_z(t)x(t)}{\sum_{z=i,(j)}^{j,(i)} w_z} \right) \qquad (3)$$

where i,j∈ { 1,2,...,m}, and each linear component of $A_z(t)x(t)$ is called subsystem. Thus, if $w_z \in [0,1]$, then we can write (2) as

$$A_z(t) = (w_i A_i + (1 - w_i)A_j) \qquad (4)$$

thence, the pencil of matrices is

$$\sigma_w [A_i, A_j] = A_z(t) \qquad (5)$$

Definition 2 *A switched fuzzy system (SFS)* Σ_{A_z} *is an autonomous time-varying system defined by*

$$\dot{x}(t) = A_z(t)x(t) \qquad (6)$$

where $A_z(t)$ will be is considered continuous for all t.

Definition 3 *A fuzzy switching action (FSA) is defined like the change of dynamic between two linear autonomous time-invariant systems* Σ_{A_i} *and* Σ_{A_j} *who are components of* Σ_{A_z}.

With the FSA, we can jump between the fuzzy sets of Σ_{A_i}, $i = \{1, 2, ..., m\}$ in the phase plan $\mathbb{R}^{n \times n}$. Then a FSA can be represented by the equation (2).

Definition 4 *A fuzzy subset* $\mathcal{FC}$ *of* $\mathbb{R}^n$ *is said to be convex if*

$$\left(\frac{w_i A_i x + w_j A_j}{w_i + w_j} \right) \in \mathcal{FC} \qquad (7)$$

whenever $\{A_i, A_j\} \in \mathcal{FC}$, $\{w_i, w_j\} \subset w$, and $w \in [0,1]$. The convex fuzzy sets have to contain, along with any two distinct matrices A_i or A_j, a certain portion of the line trough A_i and A_j, thus

$$\left\{ \frac{w_i A_i + w_j A_j}{w_i + w_j} \,\middle|\, w \in [0,1] \right\}$$

3 Stability conditions

We will consider under the assumption that all the individual subsystems are asymptotically stables. Basically, we will find that the stability is ensured if the FSA is well proposed. These conditions are in terms of convex linear combinations of matrices A_i and A_j.

An important problem is to determine necessary and sufficient conditions for the existence of a quadratic Lyapunov function $V(x) = x^T Px$, $P = P^T > 0$, such that its time derivative along any trajectory of the system (2) is negative definite, or alternatively that,

$$A_i^T P + PA_i = -Q_i \tag{8}$$

where the matrices Q_i are symmetric and positive definite. The existence of such Lyapunov function is sufficient to guarantee the uniform asymptotic stability of switching system (2).

However, the conclusions we reached about the stable equilibrium of the system can also be reached by using energy concepts.

In the following discussion we attempt to determine a common Lyapunov function of the form

$$V(x) = x^T Px \tag{9}$$

where $x \in \mathbb{R}^{n \times n}$.

In this section, necessary and sufficient conditions are derived in order to assure the existence of a common quadratic Lyapunov function for a finite number of n order linear time-invariant systems.

Lemma 1 *If P is a positive definite common matrix, such that*

$$A_i^T P + PA_i < 0 \quad \text{and} \quad A_j^T P + PA_j < 0, \tag{10}$$

where $A_i, A_j, P \in R^{n \times n}$, then from (2), we can write

$$\left(\frac{w_i A_i + w_j A_j}{w_i + w_j}\right)^T P + P\left(\frac{w_i A_i + w_j A_j}{w_i + w_j}\right) < 0 \tag{11}$$

$\forall w \in [0, 1]$.

Proof: since P is a positive definite matrix, and $\left(\frac{w_i A_i + w_j A_j}{w_i + w_j}\right) < 0$, $\forall w \in [0, 1]$, so (11) is fulfilled.

Is well know that if the Lyapunov's condition

$$A_z^T P + PA_z = -Q \tag{12}$$

is fulfilled, the switched system $\sum_{A_*}$ is stable for some $P = P^T > 0$.

Theorem 2 *A matrix A_z is Hurwitz if and only if for every nonzero vector x there exist $P = P^T > 0$ such that*

$$x^T \left(A_z^T P + PA_z\right) x < 0 \tag{13}$$

Proof: If A_z is Hurwitz $\forall w \in [0,1]$, then (12)⇒(13) otherwise (13) is violated if x is an eigenvector corresponding to an unstable eigenvalue of A_z.

Theorem 3 *The equilibrium of a fuzzy switched system*

$$\dot{x}(t) = \left(\frac{w_i A_i + w_j A_j}{w_i + w_j}\right) x(t) \tag{14}$$

is globally asymptotically stable if there exists a common positive definite matrix P for all the subsystems such that

$$A_z^T P + PA_z < 0 \tag{15}$$

Proof: Consider the scalar function $V(x(t))$ such that $V(x(t)) = x^T(t)Px(t)$, where P is a positive definite matrix, then

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\
&= x(t)^T \left(A_z^T P + PA_z\right) x(t) \\
&= x(t)^T \left(\left(\tfrac{w_i A_i + w_j A_j}{w_i + w_j}\right)^T P + P\left(\tfrac{w_i A_i + w_j A_j}{w_i + w_j}\right) \right) x(t) \\
&= x(t)^T \left(\tfrac{1}{w_i + w_j} \left(w_i A_i^T P + w_j A_j^T P + w_i P A_i + w_j P A_j\right) \right) x(t) \\
&= \tfrac{1}{w_i + w_j} \left(x(t)^T \left(w_i \left(A_i^T P + PA_i\right) + w_j \left(A_j^T P + PA_j\right) \right) x(t)\right)
\end{aligned}$$

where $w_i, w_j \in [0,1]$, for $i, j \in \{1, 2, ..., m\}$.

From Lemma 1 and equation (15), we obtain

$$\dot{V}(x(t)) < 0$$

Then, $V(x(t))$ is a Lyapunov function and the fuzzy switched system (14) is globally asymptotically stable.

Theorem 4 *Let $\Sigma_A = \{\Sigma_{A_1}, \Sigma_{A_2}, ..., \Sigma_{A_n}\}$ be a set of stable systems with a CQLF. Since the matrix A_z is composed by pairs of elements of Σ_A, then A_z is Hurwitz and Σ_{A_z} is stable $\forall w_k \in [0,1]$ for any fuzzy switching sequence.*

Proof: Since $V(x)$ is a QLF for Σ_{A_1}, $\Sigma_{A_2}$and Σ_{A_n}, then $V(x)$ will be a QLF for Σ_{A_z}. Then Σ_{A_z} is stable, and the matrix A_z is Hurwitz $\forall w \in [0,1]$ then, $V(x)$ is a CQLF. If A_z is not Hurwitz for any w, then a CQLF can not exists for Σ_{A_1}, $\Sigma_{A_2}$and Σ_{A_n}, thus a necessary condition for the existence of a CQLF for Σ_{A_1}, $\Sigma_{A_2}$and Σ_{A_n}, is that the matrix A_z be Hurwitz $\forall w$.

Theorem 5 *Let A_i and A_j be two Hurwitz matrices in $\mathbb{R}^{2\times 2}$. A necessary and sufficient condition for the dynamic systems Σ_{A_i} and Σ_{A_j} to have a quadratic Lyapunov function that the matrices A_iA_j and $A_iA_j^{-1}$ do not have real negative eigenvalues. An equivalent condition is that the pencils $\sigma_w[A_i, A_j]$, $\sigma_w\left[A_i, A_j^{-1}\right]$ are both Hurwitz for $w \in [0,1]$.*

Proof: The necessity can be directly obtained by theorem 3. The implication is showing that a sufficient condition for the matrix pencil $\sigma_w[A_i, A_j]$ to be Hurwitz is that both A_i and A_j are Hurwitz, and that the matrix $A_iA_j^{-1}$ has no real negative eigenvalues [17], [14].

The matrix pencil $\sigma_w[A_i, A_j]$ has the next characteristic polynomial

$$\det(\lambda I - \sigma_w[A_i, A_j]) = \lambda^2 - \lambda(trace(\sigma_w[A_i, A_j])) + \det(\sigma_w[A_i, A_j]) = 0$$

The pencil fulfill the Hurwitz conditions for all $w \in [0,1]$ if and only if:

a) $trace(\sigma_w[A_i, A_j]) < 0$ and

b) $det(\sigma_w[A_i, A_j]) > 0$.

It follows that the condition *a)* is satisfied if both A_i and A_j are Hurwitz. Thus for *b)*, we have that the equation for the determinant can be written as

$$\begin{aligned}\det(\sigma_w[A_i, A_j]) &= \det(w_iA_i + (1-w_i)A_j)\\ &= w_i^2\det\left(\gamma I + A_iA_j^{-1}\right)\det(A_j) \qquad (16)\end{aligned}$$

with $\gamma = (1-w_i)/w_i$ for $w_i \in [0,1]$. Due to A_j is Hurwitz, then the eigenvalues of $\det(A_j)$ are negative so, $\det(A_j)$ is positive. In order to know the eigenvalues for $\left(\gamma I + A_iA_j^{-1}\right)$, we set $\vartheta = A_iA_j^{-1}$,

$$\det(\gamma I + \vartheta) = \gamma^2 + \gamma(trace(\vartheta)) + \det(\vartheta) = 0$$

The conditions to accomplish for the latter equation is that the eigenvalues have no real negative parts, then

$$trace(\vartheta) > 0 \quad \text{and} \quad det(\vartheta) > 0 \qquad (17)$$

by determinants, we will analyze the conditions for ϑ

$$\det(\lambda I - \vartheta) = \lambda^2 - \lambda(trace(\vartheta)) + \det(\vartheta) = 0 \qquad (18)$$

therefore by Routh theorem [6], we can see that the equation given above only accept positive eigenvalues, in fact we can establish differents conditions by Hurwitz theorem, Descartes' sign rule, etc., [6]. Then from (18), we have $trace(\vartheta) > 0$ and $det(\vartheta) > 0$, then (18) is satisfied and $\det(\gamma I + \vartheta) > 0$. Finally, the eigenvalues of $\det\left(\gamma I + A_i A_j^{-1}\right)$ are positive, then are positive also for (16), so the condition *b*) is accomplished; in other words the matrix $A_i A_j^{-1}$, has no real negative eigenvalues.

Remark 1 *In the last proof, we utilise the condition (17), but is easy to see that we can analyse some others differents cases, for example:*

- $trace(\vartheta) > 0$ *and* $\det(\vartheta) > 0$ *and* $|\det(\vartheta)| > |trace(\vartheta)|$,
- $trace(\vartheta) > 0$ *and* $\det(\vartheta) < 0$ *and* $|\det(\vartheta)| < |trace(\vartheta)|$,
- $trace(\vartheta) < 0$ *and* $\det(\vartheta) < 0$ *and* $|\det(\vartheta)| + |trace(\vartheta)| < \gamma$,

etc...

The next theorem is an extension of the above result, where the matrices are in $\mathbb{R}^{n\times n}$.

Theorem 6 *Let A_i and A_j be two Hurwitz matrices in $\mathbb{R}^{n\times n}$. A necessary and sufficient condition for the dynamic systems Σ_{A_i} and Σ_{A_j} to have a quadratic Lyapunov function is that the matrices $A_i A_j$ and $A_i A_j^{-1}$ do not have real negative eigenvalues. An equivalent condition is that the pencils $\sigma_w[A_i, A_j]$, $\sigma_w\left[A_i, A_j^{-1}\right]$ are both Hurwitz for $w \in [0,1]$.*

Proof: In this proof, the necessity is in a similar way as the latter. So, for the necessity, we have that the conditions of equation (16), show that A_j is Hurwitz, then $sign(\det(A_j)) = sign(\sigma_w[A_i, A_j]) = (-1)^n$, if

$$\det(\gamma I + \vartheta) > 0 \tag{19}$$

$$\begin{aligned}\det(\gamma I + \vartheta) &= \gamma^n \\ &+\gamma^{n-1}\sum_{i=1}^{n} g_{ii} \\ &+\gamma^{n-2}\sum_{i=1}^{n}\sum_{j<i}\begin{vmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj}\end{vmatrix}\end{aligned}$$

$$+\gamma^{n-3}\sum_{i=1}^{n}\sum_{i<j}\sum_{i<j<k}\begin{vmatrix} g_{ii} & g_{ij} & g_{ik} \\ g_{ji} & g_{jj} & g_{jk} \\ g_{ki} & g_{kj} & g_{kk} \end{vmatrix}$$

$$\vdots$$

$$+|\vartheta|$$

where $\vartheta = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}$. We know that $\det(\lambda I + \vartheta) = \prod_{i=1}^{n}(\lambda - \lambda_i)$,

then we can replace the determinants by the product of eigenvalues, so

$$\begin{array}{lcl} \det(\lambda I - \vartheta) = \lambda^n & & \lambda^n \\ -\lambda^{n-1}\sum_{i=1}^{n} g_{ii} & & -\lambda^{n-1}\sum_{i=1}^{n}\lambda i \\ +\lambda^{n-2}\sum_{i=1}^{n}\sum_{j<i}\begin{vmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{vmatrix} & = & +\lambda^{n-2}\sum_{i_1<i_2<\ldots<i_{n-2}}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_{n-2}} \\ -\lambda^{n-3}\sum_{i=1}^{n}\sum_{i<j}\sum_{i<j<k}\begin{vmatrix} g_{ii} & g_{ij} & g_{ik} \\ g_{ji} & g_{jj} & g_{jk} \\ g_{ki} & g_{kj} & g_{kk} \end{vmatrix} & & -\lambda^{n-3}\sum_{i_1<i_2<\ldots<i_{n-3}}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_{n-3}} \\ \vdots & & \vdots \\ \pm|\vartheta| & & \pm\lambda_1\lambda_2\ldots\lambda_n \end{array} \quad (20)$$

then, if we apply the next equality

$$\det\left(\sigma_w[A_i, A_j]^{*i}\right) = \det\left(w_i A_i^{*i} + (1 - w_i)A_j^{*i}\right) = \det\left(A_j^{*i}\right)\det\left(\gamma I + \vartheta^{*i}\right) \quad (21)$$

where, $i = \{1, 2, ..., n\}$ and, $*i$ is the $i - th$ cofactor of det$(\cdot)$. We can visualize the eigenvalues by cofactors, thus if all the cofactors of $\det\left(A_i A_j^{-1}\right)^{*i} > 0$, so $\det\left(\gamma I + \vartheta^{*i}\right) > 0$ therefore *sign* $\left(\det\left(A_j^{*i}\right)\right)$ is equal to *sign* $\left(\det\left(\sigma_w[A_i, A_j]^{*i}\right)\right)$. Finally as A_j is Hurwitz, then the pencil $\left(\sigma_w[A_i, A_j]^{*i}\right)$ is Hurwitz.

Remark 2 *As well as we seen in remark (1), in this proof the possibility of combination of cases to analyse, is augmented due to the quantity of implied variables.*

Corollary 7 *If A_i and A_j are two Hurwitz matrices then A_z is Hurwitz* $\forall w_k \in [0, 1]$

Proof: By equation (2), we have a displacement of eigenvalues governed by the values of w. We have a movement from the eigenvalues of A_i to them of A_j and viceversa. For example and with loss of generality, if w has a variation from $w_i = 1$, $w_j = 0$ to $w_i = 0$, $w_j = 1$, the system A_z has a displacement form the eigenvalues of A_i to them of A_j, therefore A_z keep up the eigenvalues with negative real parts. Hence A_z is also Hurwitz $\forall w_k \in [0, 1]$.

Theorem 8 *A necessary and sufficient condition for the existence of a CQLF for the systems A_i and A_j, is that A_z be Hurwitz $\forall w_k \in [0, 1]$.*

Proof:

($\Rightarrow$) Let a $V(x)$ be a CQLF for A_i and A_j then, there exists a matrix $P = P^T > 0$ such that:

$$x^T \left[\left(\frac{w_i A_i + w_j A_j}{w_i + w_j} \right)^T P + P \left(\frac{w_i A_i + w_j A_j}{w_i + w_j} \right) \right] x < 0$$

for all $x \in \mathbb{R}^n$ and $w \in [0, 1]$. However we know from definition 3, that a FSA is a change of the dynamic of Σ_{A_i} to the dynamic of Σ_{A_j} and, in addition if a CQLF exists, the stability conditions in sense Lyapunov are preserved in the switching action. So, for the switching sequences where Σ_{A_i} and Σ_{A_j} are involved, the stability conditions are preserved. Therefore we can say that Σ_{A_z}, is stable for all switching sequence with Σ_{A_i} and Σ_{A_j}, as components.

($\Leftarrow$) Let A_z be not Hurwitz then, all FSA carries the Σ_{A_z} to a unstable dynamique. So that, either A_i or A_j or both are unstables then, a P matrix where:

$$\left(\frac{w_i A_i + w_j A_j}{w_i + w_j} \right)^T P + P \left(\frac{w_i A_i + w_j A_j}{w_i + w_j} \right) < 0$$

$P = P^T > 0$ does not exists, hence Σ_{A_z} is unstable, therefore $V_z(x)$ does not exists, so a common P does not exists and, finally a CQLF for the dual switching system where A_i and A_j are involved, does not exists.

4 Conclusions

In this paper, we give necessary and sufficient conditions for assure a convex solution in order to find a CQLF in a set A (see equation (1)). This convex solution is established for Hurwitz matrices and eigenvalues. A stable solution is guarantee for a switched dynamic who belong to an interval established by Hurwitz matrices in its extremes.

The above arguments formally show, that the domain into convex solutions where we can find a quadratic Lyapunov function, has a derivative negative definite along the trajectories of the system, $\dot{V}(x) < 0$.

We have seen that a switched system might become stable for fuzzy switching signals. One way to address this problem is to make sure that the interval between fuzzy switching actions is convex.

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